Indian Statistical Institute
Mid-Semestral Examination
Algebra I
2017-2018

Max Marks: 100 Time: 3 hours.

Answer all questions.

1. State true or false. Justify your answers.
(i) There exists a non-trivial homomorphism $\phi : \mathbb{Z}/5\mathbb{Z} \longrightarrow S_4$.
(ii) Every group of order 9 is abelian.
(iii) $Z(S_n)$, the center of the symmetric group $S_n$, is trivial for $n \geq 3$.
(iv) $A_4$ is a simple group. [$5 \times 4$]

2. (a) Give an example of a non-abelian group $G$, all of whose subgroups are normal in $G$.
(b) Show that $\mathbb{R}/\mathbb{Z}$ is isomorphic to $S^1 = \{z \in \mathbb{C} : |z| = 1\}$.
(c) Under the above isomorphism, what is the image of $\mathbb{Q}/\mathbb{Z}$? [8+6+6]

3. (a) Prove that two elements of $S_n$ are conjugate if and only if they have the same cycle type.
(b) Determine the elements of $C_{S_7}(\sigma)$, where $\sigma = (2\ 4\ 6)$. [10+10]

4. (a) Describe $Inn(G)$, the group of *inner automorphisms* of a group $G$. Show that
$$G/Z(G) \cong Inn(G),$$
where $Z(G)$ is the center of $G$.
(b) Let $G$ be a cyclic group of order $n$. Describe the group of automorphisms of $G$. [10+10]

5. (a) State and prove Cauchy's theorem for finite abelian groups.
(b) Using class-equation, or otherwise, prove Cauchy's theorem for finite non-abelian groups. [10+10]